ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Practical work in mathematics

Yoneda Lemma

Authors :
Thomas Rüd
Jean-Claude Ton

Supervisor :
Ms. Martina Rovelli

10^{th} May 2013

Contents

Abstract

The category theory is a mathematical domain, which aims to unify various structural concepts from different fields and provide general concepts. One of its fundamental theorems is the Yoneda Lemma, named after the mathematician Nobuo Yoneda. While the proof of the lemma is not difficult to understand, its consequences in a diversitiy of areas can not be overstated. It provides insight and important applications in other areas, in fact an algebraic version is known as Cayley's theorem.
The aim of this project is to enable a reader without or with very little knowledge about category theory to understand the Yoneda lemma and its proof. For this purpose we will provide the basic knowledge of category theory, which will be more explicitly explained by giving several examples that have been covered in other subjects such as algebra and topology. Finally, we are going to illustrate one of its applications on simplicial sets.

1 Introduction

"Chaque objet abstrait est devenu concret par l'usage [...] un objet concret est un objet abstrait auquel on a fini par s'habituer." Laurent Schwartz

Introduced in 1945 by Eilenberg and Mac Lane in a paper entitled "General Theory of Natural Equivalences", category theory has grown over the past few decades into a branch in mathematics, like algebra and analysis. Roughly speaking, it is a general mathematical theory of structures, which provides numerous abstract concepts of concrete ones in diverse domains in mathematics.
The objective of this paper is to provide the reader with the necessary knowledge in category theory in order to understand the lemma. For this purpose, we are going to introduce some definitions e.g. categories in chapter 2, where we provide a generous amount of accessible examples such as the categories of groups or topological spaces. Similarly, we will illustrate other important notions like functors and natural transformations in the same section.
After setting up the right framework, the reader should be able to understand what will follow.

In chapter 3, we are going to overcome the first milestone in category theory by approaching the Yoneda Lemma. To give the reader an intuitive interpretation of the lemma, Ravi Vakil, an algebraic geometry teacher at Stanford University once explained the Yoneda Lemma as follows: *"You work at a particle accelerator. You want to understand some particle. All you can do is throw other particles at it and see what happens. If you understand how your mystery particle responds to all possible test particles at all possible test energies, then you know everything there is to know about your mystery particle."*
The proof is divided in two parts: in section 3.1 we show that we have in fact a bijection and in the following one we analyze the "good behaviour" of this bijection by changing one of the studied objects. In this process, we consider the Yoneda embedding functor, and observe which information it holds for a given object.

The last chapter focuses on providing a concrete application of the Yoneda Lemma. Furthermore, we want to emphasize how to use the categorical language we previously intruduced, in algebraic topology. In particular, we first consider a special kind of topological spaces, the simplicial complexes, made by attaching in a nice way topological simplices. Then we generalize the simplicial complexes by intruducing the simplicial sets, that will become useful after applying the Yoneda Lemma. In order to return to some topological spaces we finish with the notion of singular sets.

2 Category Theory

2.1 Axiomatic foundations

In this section, we will set some preliminary logic foundations, before we start our discussion about categories. We assume the standard Zermelo-Fraenkel axioms for the set theory and the existence of a set U which is the universe.
The Zermelo-Fraenkel axioms (on a membership relation $\in$) are the following:

- Extensionality: sets with the same elements are equal;
- Null set: there exists a set with no elements;
- Existence of the sets $\{u, v\}$, $\langle u, v\rangle$, $\mathcal{P}u$ and $\bigcup x$ for all sets u, v, and x, where we write $\{\ \}$ as a set, $\langle\ ,\ \rangle$ as an ordered pair, $\mathcal{P}$ as a power set of a set, and $\bigcup$ as a union;
- Infinity: the axiom of infinity holds;
- Choice: the axiom of choice holds;
- Regularity: every non-empty set A contains an element B which is disjoint from A;
- Replacement: the image of a set under a *function* is a set. More precisely, let a be a set and $\varphi(x, y)$ a property which is functional for x in a, in the sense that $\varphi(x, y)$ and $\varphi(x, y')$ for $x \in a$ imply $y = y'$, and that for each $x \in a$ there exists a y with $\varphi(x, y)$; then there exists a set consisting of all those y such that $\varphi(x, y)$ holds for $x \in a$.

Moreover, the universe U is defined by the following axiom:

- $[x \in y$ and $y \in U] \Rightarrow x \in U$;
- $[I \in U$ and $\forall i \in I\ x_i \in U] \Rightarrow \bigcup_{i\in I} x_i \in U$;
- $[x \in U] \Rightarrow \mathcal{P}(x) \in U$;
- $[x \in U$ and $f : x \to y$ surjective funtion $] \Rightarrow y \in U$.
- $\mathbb{N} \in U$

We call **small sets** the elements of U.

2.2 Definitions and basic notions

In this section, we will define the elementary notions in order to understand the Yoneda Lemma.

2.2.1 Categories

Definition 1 (Category)**.** A **category** $\mathcal{C}$ consists of the following:

1. a set $|\mathcal{C}|$, whose elements will be called **objects** of the category;
2. for every pair A, B of objects, a set $\mathcal{C}$(A,B) or $\mathrm{Hom}_{\mathcal{C}}(A, B)$, whose elements will be called **morphisms** or **arrows** from A to B;
3. for every triple A, B, C of objects, a **composition law**;

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C);$$

   the composite of the pair (f, g) will be written $g \circ f$ or just gf;
4. for every object A, a morphism $1_A \in \mathcal{C}(A, A)$, called the **identity** on A.

Such that the following axioms are satisfied:

1. Associativity axiom: given morphisms $f \in \mathcal{C}(A, B),\ g \in \mathcal{C}(B, C),\ h \in \mathcal{C}(C, D)$ the following equality holds:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. Identity axiom: given morphisms $f \in \mathcal{C}(A, B)$ the following equalities hold:

$$f = 1_B \circ f = f \circ 1_A.$$

Definition 2 (small category and locally small category)**.** A category $\mathcal{C}$ is called **locally small** if each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a small set for $A, B \in |\mathcal{C}|$. Furthermore, if the collection of objects is a small set, then we say that $\mathcal{C}$ a **small category**.

Examples :
Let's introduce some important examples of categories.

- Category of sets (**Set**):
  The objects of this category are small sets and the arrows the functions between them. The composition law is the usual function composition $\circ$ and for any set S, 1_S is the usual identity function.
  From set theory we know that the function composition is associative and the identity functions are the neutral elements.

- Category of groups (**Grp**):
  The objects of this category are all groups, the arrows are group homomorphisms. The composition law is the usual function composition $\circ$ and the identity is the identity function. Since the axioms already hold at the level of functions, we conclude that **Grp** is a category.

We can similarly consider the category of vector spaces over a fixed field K, denoted by $\mathbf{Vect}_K$, the objects are all those vector spaces, and the arrows are linear functions, with the usual identity and composition. In the same way, we have the category of commutative rings **Rng**; the objects are commutative rings, and the arrows are ring homomorphisms.

- Category of topological spaces (**Top**):
  The objects of this category are all topological spaces and the arrows are continuous mappings. Again, the composition law is the usual function composition $\circ$ and the neutral element is the identity function. The axioms are also directly inherited from the mapping properties, because the identity is always continuous in a topological space and the composition of continuous function is continuous as well.

- Categories of small categories (**Cat**):
  The objects of this category are all small categories and the morphisms are functors (see next section) between small categories. The composition of morphisms in **Cat** is the functor composition and the identity functor acts as the identity morphism. The composition on functors is well defined and works componentwise.

- Opposite category ($\mathcal{C}^{\mathbf{op}}$):

  Given a category $\mathcal{C}$ we can define its opposite category $\mathcal{C}^{\mathbf{op}}$. The objects are the same as for $\mathcal{C}$ i.e. $|\mathcal{C}^{\mathbf{op}}| = |\mathcal{C}|$ and for each A, B $\mathcal{C}^{\mathbf{op}}(A, B) = \mathcal{C}(B, A)$. If f is an arrow of $\mathcal{C}$, we write $f^{\mathbf{op}}$ when we want to look at its equivalent in the opposite category. Given $f^{\mathbf{op}} \in \mathcal{C}^{\mathbf{op}}(A, B)$, $g^{\mathbf{op}} \in \mathcal{C}^{\mathbf{op}}(B, C)$, ($f$, g are the equivalent arrows in $\mathcal{C}$) the composition law is given by $g^{\mathbf{op}} \circ^{\mathbf{op}} f^{\mathbf{op}} = (f \circ g)^{\mathbf{op}}$ where $\circ$ is the composition law from $\mathcal{C}$. The neutral element is the identity function. The axioms hold because $\mathcal{C}$ is a category.

Remark 1. *All of the categories of sets with structures are locally small.*

2.2.2 Functors

Definition 3 (Functor)**.** A **functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of the following:

1. a mapping
   $$|\mathcal{A}| \to |\mathcal{B}|$$
   between the sets of objects of $\mathcal{A}$ and $\mathcal{B}$; the image of $A \in \mathcal{A}$ is written $F(A)$ or just FA;

2. for every pair of objects A, A' of $\mathcal{A}$, a mapping
   $$\mathcal{A}(A, A') \to \mathcal{B}(FA, FA');$$
   the image of $f \in \mathcal{A}(A, A')$ is written $F(f)$ or just Ff.

A functor must obey the following axioms:

1. for every pair of morphism $f \in \mathcal{A}(A, A'),\ g \in \mathcal{A}(A', A'')$

$$F(g \circ f) = F(g) \circ F(f);$$

2. for every object $A \in \mathcal{A}$

$$F(1_A) = 1_{FA}.$$

Examples :

- Maybe the most obvious functor to create is the identity functor. Given a category $\mathcal{C}$ we define:

$$\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

$$\mathrm{Id}_{\mathcal{C}}(C) = C, \quad \mathrm{Id}_{\mathcal{C}}(f) = f.$$

  We can quickly check the two axioms:

  1. let $f \in \mathcal{C}(A, B),\ g \in \mathcal{C}(B, C)$ $\mathrm{Id}_{\mathcal{C}}(g \circ f) = g \circ f = \mathrm{Id}_{\mathcal{C}}(g) \circ \mathrm{Id}_{\mathcal{C}}(f)$;
  2. let $C \in |\mathcal{C}|$, $\mathrm{Id}_{\mathcal{C}}(1_C) = 1_C$.

- If we consider a locally small category $\mathcal{C}$ and $C \in |\mathcal{C}|$, we can define a functor called a **representable functor** (which we will use quite often later):

$$\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$$

$$\mathcal{C}(C, -)(A) = \mathcal{C}(C, A),$$

  if $f \in \mathcal{C}(A, B)$,

$$\mathcal{C}(C, -)(f) = \mathcal{C}(C, f) : \mathcal{C}(C, A) \to \mathcal{C}(C, B)$$

$$\mathcal{C}(C, f)(g) = f \circ g.$$

  Let's check the two axioms:

  1. let $f \in \mathcal{C}(A, B),\ g \in \mathcal{C}(B, C),\ h \in \mathcal{C}(C, A)$, $\mathcal{C}(C, -)(g \circ f)(h) = \mathcal{C}(C, g \circ f)(h) = g \circ f \circ h = \mathcal{C}(C, g)(f \circ h) = \mathcal{C}(C, g) \circ \mathcal{C}(C, f)(h)$;
  2. let $f \in \mathcal{C}(C, A)$, $\mathcal{C}(C, 1_A)(f) = 1_A \circ f$ so $\mathcal{C}(C, 1_A) = 1_{\mathcal{C}(C,A)}$.

- Given two categories $\mathcal{C}$ and $\mathcal{D}$, and $D \in |\mathcal{D}|$, we can define a **constant functor to D**:

$$\Delta_D : \mathcal{C} \to \mathcal{D}$$

$$\Delta_D(A) = B \quad \Delta_D(f) = 1_B.$$

  The axioms are clearly satisfied here.

- We can define functors on all categories of structured sets ($\mathbf{Vect}_K$, **Grp**, **Rng**, **Top**, $\cdots$) such as the **forgetful functors**, that forgets the structure of these objects. For example:

$$F : \mathbf{Grp} \to \mathbf{Set}, \text{ defined as follows;}$$

for any group $(G, +)$ and group homomorphism $f : (A, +_A) \to (B, +_B)$

$$F((G, +)) = G$$

$$F(f) = \bar{f},$$

where $\bar{f} : A \to B$ is the same as f thought as a function, we just forget it is a homomorphism. The axioms easily hold.

Similarly, we can also define a forgetful functor from **Top** to **Set** that forgets the topology of spaces and forgets the continuity of functions.

- We can also add a structure to a set, for example we can define the functor:

  $$F_{disc} : \mathbf{Set} \to \mathbf{Top}, \text{ defined by}$$

  $$F(X) = (X, \tau_{disc}) \quad F(f) = f$$

  where τ_{disc} is the discrete topology, i.e. $\tau_{disc} = \mathcal{P}(X)$.

- We define two functors from **Rng** to **Grp**.

  The first one is GL_n; to every commutative ring R of $|\mathbf{Rng}|$ it associates the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices with coefficients in R. To each ring homomorphism $f : R \to R'$ it associates the group homomorphism $\mathrm{GL}_n(f) : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R')$, such that $(\mathrm{GL}_n(f)(M))_{i,j} = f(M_{i,j})$.

  The second one is $(-)^*$, to each commutative ring $(R, +, .)$ of $|\mathbf{Rng}|$ it associates $(R^*, .)$ the group of invertible elements of R, and to each ring homomorphism $f : R \to R'$ the group morphism $f^* : R^* \to R'^*$ such that $f^*(x) = f(x)$.

  In both cases, the functorial properties easily hold.

Definition 4 (Contravariant functor)**.** A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathbf{op}}$ to $\mathcal{D}$.
Functors that are not contravariant (the ones we previously introduced) are said to be **covariant**.

Example :

- We can also define a contravariant representable functor, as follows. Let $\mathcal{C}$ be a locally small category and $C \in |\mathcal{C}|$. We define

  $$\mathcal{C}(-, C) : \mathcal{C}^{\mathbf{op}} \to \mathbf{Set}$$

  $$\mathcal{C}(-, C)(A) = \mathcal{C}(A, C),$$

  and if $f \in \mathcal{C}(A, B)$,

  $$\mathcal{C}(-, C)(f) = \mathcal{C}(f, C) : \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$

  $$\mathcal{C}(f, C)(g) = g \circ f.$$

2.2.3 Natural Transformations

Definition 5 (Natural Transformation)**.** Consider two functors $F,\ G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ from a category $\mathcal{A}$ to a category $\mathcal{B}$. A **natural transformation** $\alpha : F \Rightarrow G$ from F to G is a family of morphisms $(\alpha_A :\ FA \to GA)_{A \in \mathcal{A}}$ of $\mathcal{B}$ indexed by the objects of $\mathcal{A}$ such that for every morphism $f :\ A \to A'$ in $\mathcal{A}$, $\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A$, i.e. the follwing diagram commutes.

$$
\begin{array}{ccccc}
A & \quad & FA & \xrightarrow{\alpha_A} & GA \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\
A' & & FA' & \xrightarrow[\alpha_{A'}]{} & GA'
\end{array}
$$

Let F, G and H be functors from $\mathcal{A}$ to $\mathcal{B}$ and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ are natural transformations, the composition law is as follows,

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

One can see that this composition law is associative and for each functor F it has a neutral element, which is simply the natural transformation Id_F whose A-component is Id_{FA}.

Remark 2. *Let $\mathcal{A}, \mathcal{B}$ be categories. We write*$\mathbf{Fun}(\mathcal{A}, \mathcal{B})$ *as the category of functors from $\mathcal{A}$ to $\mathcal{B}$, where the arrows are natural transformations between them. Equivalently we write* $\mathbf{Fun}^*(\mathcal{A}, \mathcal{B})$ *the category of covariant functors from $\mathcal{A}$ to $\mathcal{B}$.*

Examples :

- We use now the representable functor we previously defined (see section 2.2.2). A natural transformation that we will use later is built by taking a category $\mathcal{A}$ and a morphism $f \in \mathcal{A}(A, B)$. We can create a natural transformation $\mathcal{A}(f, -) : \mathcal{A}(B, -) \Rightarrow \mathcal{A}(A, -)$. This transformation is defined, for $C \in |\mathcal{A}|$ and $g \in \mathcal{A}(B, C)$ by

  $$\mathcal{A}(f, -)_C(g) = \mathcal{A}(f, C)(g) = g \circ f.$$

  We can check that it is a natural transformation, i.e. for each $g \in \mathcal{A}(C, D)$ the following diagram commutes

  $$
  \begin{array}{ccccc}
  C & \quad & \mathcal{A}(B, C) & \xrightarrow{\mathcal{A}(f, C)} & \mathcal{A}(A, C) \\
  {\scriptstyle g}\big\downarrow & & {\scriptstyle \mathcal{A}(B, g)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{A}(A, g)} \\
  D & & \mathcal{A}(B, D) & \xrightarrow[\mathcal{A}(f, D)]{} & \mathcal{A}(A, D)
  \end{array}
  $$

In fact, we have for all $h \in \mathcal{A}(B, C)$ that

$$\begin{aligned}
(\mathcal{A}(f, -)_D \circ \mathcal{A}(B, -)(g))(h) &= (\mathcal{A}(f, D) \circ \mathcal{A}(B, g))(h) \\
&= \mathcal{A}(f, D)(g \circ h) \\
&= (g \circ h) \circ f \in \mathcal{A}(A, D) \\
&= g \circ (h \circ f) \text{ by associativity of the arrows} \\
&= g \circ \mathcal{A}(f, A)(h) \\
&= (\mathcal{A}(A, g) \circ \mathcal{A}(f, C))(h) \\
&= (\mathcal{A}(A, -)(g) \circ \mathcal{A}(f, -)_C)(h).
\end{aligned}$$

Hence, the transformation is natural.

- Let's recall that we defined two functors $\mathrm{GL}_n, (-)^* : \mathbf{Rng} \to \mathbf{Grp}$ (see 2.2.2). We will build a natural transfomation between them, called determinant defined by

  $$\det{}_R : (\mathrm{GL}_n(R)) \to R^* \quad : M \mapsto \det(M)$$

  It is well defined, because if $M \in \mathrm{GL}_n(R)$ is an invertible matrix, its determinant is invertible in R, so $\det(M) \in R^*$.

  Let's prove that this transformation is natural, we need to show that for each $f : A \to B \in \mathbf{Rng}$ the following diagram commutes.

  $$\begin{array}{ccccc}
  A & \quad & GL_n(A) & \xrightarrow{\det_A} & A^* \\
  f \big\downarrow & & GL_n(f) \big\downarrow & & \big\downarrow f^* \\
  B & & GL_n(B) & \xrightarrow[\det_B]{} & B^*
  \end{array}$$

  In other words, $\det_B \circ \mathrm{GL}_n(f) = (f)^* \circ \det_A$.

  Let $M \in \mathrm{GL}_n(A)$. We have

  $$\begin{aligned}
  \det{}_B \circ \mathrm{GL}_n(f)(M) &= \det{}_B \circ (\mathrm{GL}_n(f)(M)) \\
  &= \sum_{\sigma \in S_n} \prod_{i=1}^{n} (\mathrm{GL}_n(f)(M))_{i,\sigma(i)} \\
  &= \sum_{\sigma \in S_n} \prod_{i=1}^{n} f(M_{i,\sigma(i)}) \\
  &= \sum_{\sigma \in S_n} f\left(\prod_{i=1}^{n} M_{i,\sigma(i)}\right) \quad \text{because f is a ring homomorphism} \\
  &= f\left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} M_{i,\sigma(i)}\right) \quad \text{for the same reason}
  \end{aligned}$$

$$
\begin{aligned}
f\left(\sum_{\sigma\in S_n}\prod_{i=1}^{n} M_{i,\sigma(i)}\right) &= f^*\left(\sum_{\sigma\in S_n}\prod_{i=1}^{n} M_{i,\sigma(i)}\right)\\
&= f^*\left(\det(M)\right)\\
&= f^*\circ \det{}_A(M).
\end{aligned}
$$

Thus, det is a natural transformation.

- Canonical morphism $\sigma_v : V \to V^{**} : v \mapsto v^{**}$.
From an algebraic result, we know that each vector space V is isomorphic to its bidual V^{**}. We may show that this morphism defines a natural transformation from the identity functor to the bidual functor, for every vector space V.
Consider the identity functor $\mathrm{Id}_{Vect} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ and the bidual functor $()^{**} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$. We have $\sigma_v : V \to V^{**} : v \mapsto \mathrm{ev}_v$, where $\mathrm{ev}_v : V^* \to V^{**} : \phi \mapsto \phi(v)$. We claim that setting $\sigma := \{\sigma_v\}_{v\in Vect}$ we get a natural transformation from Id_{Vect} to $(\ )^{**}$.
In order to prove that σ is natural, we have to show that for each $f : V \to W$ the following diagram commutes.

$$
\begin{array}{ccccc}
V & \qquad & \mathrm{Id}_V & \xrightarrow{\sigma_v : v \mapsto \mathrm{ev}_v} & V^{**}\\
\Big\downarrow{\scriptstyle \mathrm{Id}_{\mathbf{Vect}} f} & & \Big\downarrow{\scriptstyle \mathrm{Id}_{\mathbf{Vect}} f} & & \Big\downarrow{\scriptstyle f^{**}}\\
W & & \mathrm{Id}_W & \xrightarrow[\sigma_w : w \mapsto \mathrm{ev}_w]{} & W^{**}
\end{array}
$$

Let $v \in V$, then:

$$(\sigma_w \circ f)(v) = \mathrm{ev}_{fv}.$$

Also, we have:

$$
\begin{aligned}
f^{**}\circ\sigma_v(v) &= f^{**}\circ ev_v\\
&= f^{**}(\phi \mapsto \phi(v))\\
&= \phi \mapsto (f\circ\phi)(v)\\
&= \phi \mapsto \phi(fv)\\
&= \mathrm{ev}_{fv}.
\end{aligned}
$$

We have shown that the diagram commutes and thus σ is natural.

3 The Yoneda Lemma

3.1 The Yoneda Bijection

Theorem 1 (The Yoneda Lemma)**.** *Consider a functor $F : \mathcal{A} \to Set$ from a locally small category $\mathcal{A}$ to the category of sets, an object $A \in \mathcal{A}$ and the corresponding* *representable functor $\mathcal{A}(A,-) : \mathcal{A} \to Set$. Then the follwing is a bijective correspondence:*

$$\theta_{F,A} : Nat(\mathcal{A}(A,-), F) \xrightarrow{\cong} FA$$
$$\theta_{F,A}(\alpha) = \alpha_A(1_A)$$

between the set of natural transformations from $\mathcal{A}(A,-)$ to F and the elements of the set FA.

Proof. Consider a given element $a \in FA$. We define, for every object B $\in \mathcal{A}$, a mapping

$$\tau(a)_B : \mathcal{A}(A,B) \longrightarrow FB,$$

given by $\tau(a)_B(f) = F(f)(a)$. Hence, this class of mappings defines a natural transformation

$$\tau(a) : \mathcal{A}(A,-) \Rightarrow F.$$

Since, for every morphism $g : B \to C$ in $\mathcal{A}$, the following relation holds.

$$Fg \circ \tau(a)_B = \tau(a)_C \circ \mathcal{A}(A,g),$$

i.e. the diagram commutes.

$$\begin{array}{ccc} \mathcal{A}(A,B) & \xrightarrow{\tau(a)_B} & FB \\ {\scriptstyle \mathcal{A}(A,g)}\downarrow & & \downarrow{\scriptstyle Fg} \\ \mathcal{A}(A,C) & \xrightarrow[\tau(a)_C]{} & FC \end{array}$$

In fact, $\forall f \in \mathcal{A}(A,B)$, by the functoriality of f we get:

$$\begin{aligned} Fg \circ \tau(a)_B(f) &= Fg(Ff(a)) \\ &= Fg \circ Ff(a) \\ &= F(g \circ f)(a) \\ &= \tau(a)_C(\mathcal{A}(A,g)(f). \end{aligned}$$

In order to finish the proof, we now have to show that $\theta_{F,A}$ and τ are the inverse of each other.
Let $a \in FA$, we have

$$\theta_{F,A}(\tau(a)) = \tau(a)_A(1_A) = (F1_A)(a) = 1_{FA}(a),$$

so $\theta_{F,A} \circ \tau = \mathrm{Id}_{\mathcal{FA}}$.
On the other hand, starting from $\alpha : \mathcal{A}(A,-) \Rightarrow F$ and choosing a morphism $f : A \to B$ in $\mathcal{A}$,

$$\begin{aligned} \tau(\theta_{F,A}(\alpha))_B(f) &= \tau(\alpha_A(1_A))_B(f) \\ &= F(f)(\alpha_A(1_A)) \\ &\overset{(*)}{=} \alpha_B(\mathcal{A}(A,f)(1_A)) \\ &= \alpha_B(f \circ 1_A) \\ &= \alpha_B(f), \end{aligned}$$

where $(*)$ follows from the naturality of α. So $\tau(\theta_{FA}(\alpha))$ and α coincide since they have the same components. $\square$

There is a contravariant form of the Yoneda lemma for contravariant functors.

Theorem 2 (contravariant Yoneda Lemma)**.** *Consider a contravariant functor $F : \mathcal{A} \to Set$ from an locally small category $\mathcal{A}$ to the category of sets, an object $A \in \mathcal{A}$ and the corresponding contravariant representable functor $\mathcal{A}(-, A) : \mathcal{A}^{\mathbf{op}} \to \mathbf{Set}$. Then the follwing is a bijective correspondence*

$$\theta_{F,A} : \mathbf{Nat}(\mathcal{A}(-, A), F) \xrightarrow{\cong} FA$$

$$\theta_{F,A}(\alpha) = \alpha_A(1_A^{\mathbf{op}})$$

between the natural transformations from $\mathcal{A}(-, A)$ to F and the elements of the set FA.

Proof. The proof of the contravariant Yoneda Lemma is similar to the covariant one. $\square$

3.1.1 Yoneda Embedding

Definition 6 (full and faithful)**.** Consider a functor $F : \mathcal{A} \to \mathcal{B}$. For every pair of objects A, $A' \in \mathcal{A}$, we have a mapping

$$\mathcal{A}(A, A') \longrightarrow \mathcal{B}(FA, FA')$$

$$f \mapsto Ff.$$

1. The functor F is said to be **faithful** when for all A, A' the abovementioned mapping is injective.
2. The functor F is said to be **full** when for all A, A' the abovementioned mappings is surjective.
3. The functor F is said to be **fully faithful** when it is both full and faithful.

The Yoneda lemma deals with the functors from a locally small category $\mathcal{A}$ to **Set**. As in Remark 2, we can collect them in a category $\mathbf{Fun}(\mathcal{A}, \mathbf{Set})$, where objects are the functors from $\mathcal{A}$ to **Set**, and the arrows are natural transformations. Our aim now is to show that this category in a certain sense contains A.
Let A be a locally small category, we define **Yoneda embedding functors**, as follows:

$$\begin{aligned}
&Y^* : \mathcal{A}^{\mathbf{op}} \longrightarrow \mathbf{Fun}(\mathcal{A}, \mathbf{Set}),\\
&Y^*(A) = \mathcal{A}(A, -), \quad Y^*(f) = \mathcal{A}(f, -)\\
&Y_* : \mathcal{A} \longrightarrow \mathbf{Fun}^*(\mathcal{A}, \mathbf{Set}),\\
&Y_*(A) = \mathcal{A}(-, A), \quad Y_*(f) = \mathcal{A}(-, f).
\end{aligned}$$

We can check that $Y^*(1_B) = 1_{Y*B}$ and also, if $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, $Y^*(g \circ f) = Y^*(f) \circ Y^*(g)$. So this is a contravariant functor. Similarly, Y_* is a covariant functor.

Now our motivation is equivalent to the following proposition:

Proposition 1 (Yoneda embedding)**.** *The Yoneda embedding functors are fully faithful.*

Proof. We prove the result for Y^*. The second case is similar. What we want to show is that for all $A, B \in |\mathcal{A}|$ the following mapping is bijective:

$$\mathcal{A}(B, A) \longrightarrow \mathbf{Fun}(\mathcal{A}, \mathbf{Set})(Y^*(A), Y^*(B)), \quad f \mapsto Y^*f.$$

Using the definition of Y^*, we can replace Y^*f with $\mathcal{A}(f, -)$ and since the arrows of $\mathbf{Fun}(\mathcal{A}, \mathbf{Set})$ are natural transformations we have that

$$\begin{aligned}\mathbf{Fun}(\mathcal{A}, \mathbf{Set})(Y^*(A), Y^*(B)) &= \mathbf{Fun}(A, \mathbf{Set})(\mathcal{A}(A, -), \mathcal{A}(B, -)) \\ &= \mathbf{Nat}(\mathcal{A}(A, -), \mathcal{A}(B, -)).\end{aligned}$$

So we can write the functor as

$$\mathcal{A}(B, A) \longrightarrow \mathbf{Nat}(\mathcal{A}(A, -), \mathcal{A}(B, -)), \quad f \mapsto \mathcal{A}(f, -)$$

We prove that this is exactly the Yoneda bijection when F is the functor $\mathcal{A}(B, -)$. Let's compute τ in this case:

$$\tau : \mathcal{A}(B, A) \to \mathbf{Nat}(\mathcal{A}(A, -), \mathcal{A}(B, -)), \quad \tau(f)_{\mathcal{A}(B,-)} : \mathcal{A}(A, C) \to \mathcal{A}(B, C)$$
$$\tau(f)_{\mathcal{A}(B,-)}(g) = \mathcal{A}(B, -)(g)(f) = \mathcal{A}(B, g)(f) = g \circ f = \mathcal{A}(f, -)_C(g).$$

Finally we found $\tau(f) = \mathcal{A}(f, -) \in \mathbf{Nat}(\mathcal{A}(A, -), \mathcal{A}(B, -))$, which is exactly the mapping we considered. □

3.2 Naturality of the Yoneda Bijection

The Yoneda lemma describes a function between $\mathbf{Nat}(\mathcal{A}(A, -), F)$ and FA where A is an object of a locally small category $\mathcal{A}$ and F is a functor from $\mathcal{A}$ to $\mathbf{Set}$. Now we want to prove that it is natural in both the variables, i.e. if we change either A with a morphism, or F with a natural transformation, the bijection is consistent.
Basically, if $F : \mathcal{A} \to \mathbf{Set}$ is a fixed functor, we define the functor $N : \mathcal{A} \to \mathbf{Set}$ as follows

$$N(A) = \mathbf{Nat}(\mathcal{A}(A, -), F)$$

$$N(f) : \left| \begin{array}{l} \mathbf{Nat}(\mathcal{A}(A, -), F) \to \mathbf{Nat}(\mathcal{A}(B, -), F) \\ \alpha \mapsto \alpha \circ \mathcal{A}(f, -) \end{array} \right. \quad \text{where } f \in \mathcal{A}(A, B).$$

So N is the composition of two representable functors. We define a natural transformation $\eta : N \Rightarrow F$ by $\eta_A = \theta_{F,A}$.

Actually, it is natural, i.e $(\theta_{F,B} \circ N(f))(\alpha) = (F(f) \circ \theta_{F,A})(\alpha)$.

$$\begin{aligned} \text{In fact } (\theta_{F,B} \circ N(f))(\alpha) &= \theta_{F,B}(\alpha \circ \mathcal{A}(f, -)) \\ &= (\alpha \circ \mathcal{A}(f, -))_B(1_B) \\ &= \alpha_B(f) \\ &= (\alpha_B \circ \mathcal{A}(A, f))(1_A) \\ &= F(f)(\alpha_A(1_A)) \\ &= (F(f) \circ \theta_{F,A})(\alpha). \end{aligned}$$

This means that the Yoneda bijection is natural in A.

Now let's fix $A \in |\mathcal{A}|$ and consider the representable functor $M : \mathbf{Fun}(\mathcal{A}, \mathbf{Set}) \to \mathbf{Set}$ with respect to $\mathcal{A}(A, -)$. More explicitly,

$$M(F) = \mathbf{Nat}(\mathcal{A}(A, -), F),$$

and given two functors $F, G : \mathcal{A} \to \mathbf{Set}$ and a natural transformation $\gamma : F \Rightarrow G$ we have
$M(\gamma) : \left| \begin{array}{l} \mathbf{Nat}(\mathcal{A}(A, -), F) \longrightarrow \mathbf{Nat}(\mathcal{A}(A, -), G) \\ \alpha \longmapsto \gamma \circ \alpha \end{array} \right.$. We remark that this functor actually takes values in *Set* thanks to the Yoneda bijection.
Moreover, consider the functor evaluation in A $\mathrm{ev}_A : \mathbf{Fun}(\mathcal{A}, \mathbf{Set}) \to \mathbf{Set}$, given by

$$\mathrm{ev}_A(F) = FA$$

$$\mathrm{ev}_A(\gamma) = \gamma_A, \ \forall F : \mathcal{A} \to \mathbf{Set}, \gamma : F \Rightarrow G$$

It is well defined thanks to the definition of the composition of natural transformation and the identity of natural transformation.
Now we can define a natural transformation $\mu : M \Rightarrow \mathrm{ev}_A$, $\mu_F = \theta_{F,A}$. In order to check that M is natural, we need to show that $(\theta_{G,A} \circ M(\gamma))(\alpha) = (\mathrm{ev}_A(\gamma) \circ \theta_{F,A})(\alpha)$.

$$\begin{aligned} \text{But } (\theta_{G,A} \circ M(\gamma))(\alpha) &= \theta_{G,A}(\gamma \circ \alpha) \\ &= (\gamma \circ \alpha)_A(1_A) \\ &= \gamma_A(\alpha_A(1_A)) \\ &= (\mathrm{ev}_A(\gamma) \circ \theta_{F,A})(\alpha). \end{aligned}$$

So we have that the Yoneda bijection is natural in F as well. □

Remark 3. *The naturality holds in the contravariant form, too.*

4 Simplicial sets: an important category of contravariant functors

Simplicial sets have many applications in algebraic topology, where they give tools to study a particular topology. We do not intend to discuss any important applications. Instead, we aim to provide an elementary introduction to simplicial sets, understandable with the very basic knowledge of category theory given in the previous chapter.

4.1 Simplicial complexes

Definition 7 (n-simplex)**.** Let $\{u_0, \ldots, u_n\}$ be points in $\mathbb{R}^N$. A point x $= \sum_{i=0}^{N} \lambda_i u_i$ is a linear combination of the u_i if the sum of all λ_i is one. The linear combination $x = \sum_{i=0}^{N} \lambda_i u_i$ is a convex combination if all λ_i are non-negative. A convex hull is the set of a convex combinations. We call a geometric **n-simplex** a convex hull spanned by n+1 linear independent points denoted by $[u_0, \cdots, u_n]$.

Definition 8 (Simplicial complex)**.** A geometric **simplicial complex** X in $\mathbb{R}^N$ consists of a collection of simplices of various dimensions, in $\mathbb{R}^N$ such that:

1. every face of a simplex of X is in X;
2. the intersection of any two simplices of X is a face of each of them.

Remark 4. *To remove redundancy, we order the complexes taking the set of vertices totally ordered such that $[u_{i_0}, \cdots, i_{i_n}]$ is a simplex of X if and only if $u_{i_j} < u_{i_l} \quad \forall j < l$.*

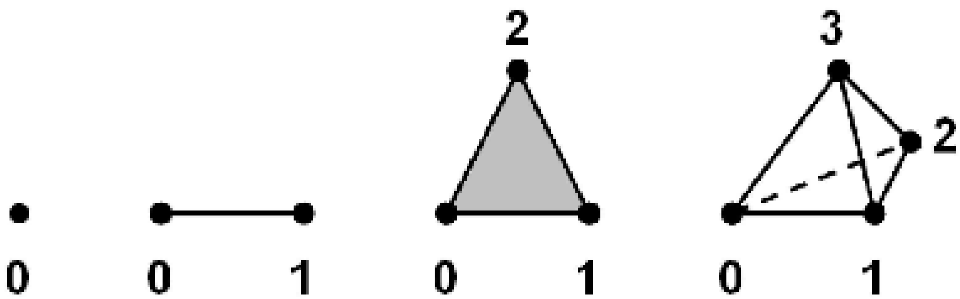

Figure 1: Example of 0-, 1-, 2-, and 3-simplices.

We want to define maps between such structures.

Definition 9 (Simplicial map)**.** If K, L are two simplicial complexes, then a **simplicial map** $f : K \to L$ is determined by taking the vertices v_i of K to the vertices $f(v_i)$ of L such that if $[u_{i_0}, \cdots, u_{i_n}]$ is a simplex of K then $f(u_{i_0}), \cdots, f(u_{i_n})$ are all vertices (not necessarily unique) of some simplex in L. The rest of the function is determined by linear interpolation. If $x = \sum_{j=1}^{n} \lambda_j u_{i_j}$ in barycentric coordinates of the simplex spanned by the v_{i_j}, then $f(x) = \sum_{j=1}^{n} \lambda_j f(u_{i_j})$. Such a function is continuous.

Definition 10 (Face of a simplex)**.** Given a n-simplex $[u_0, \cdots, u_i, \cdots, u_n]$, its *ith* **face** is the (n-1)-simplex $[u_0, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n]$.

We want to study particular maps, like the inclusion of a simplex in a complex and the collapsing of a simplex in a simplex of lesser dimension. For example, there is a simplicial map that assigns an edge to a triangle and another one that assigns a triangle to one of its edges.

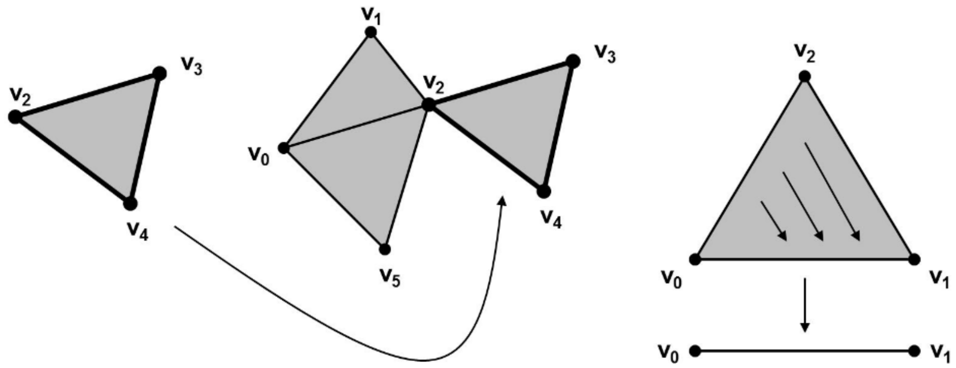

Figure 2: Inclusion of a 2-simplex in a complex (left) and collapsing of a 2-simplex in a 1-simplex (right).

4.2 Simplicial sets

Definition 11 (Category Δ)**.** The category Δ has as objects the finite ordered sets $[n] = \{0, \cdots, n\}$. The morphisms are order-preserving functions $[m] \to [n]$. The identity morphisms and the composition law are the usual functions on sets.

Remark 5. *We can define **coface** and **codegeneracy maps** (respectively d_i and s_i, $\quad n \in \mathbb{N}$) as follows:*

$$d_i \in \Delta([n],[n+1]) \quad d_i(j) = \begin{cases} j \text{ if } j < i \\ j+1 \text{ if } j \geq i \end{cases}$$

i.e. $d_i(0, \cdots, n) = (d_i(0), \cdots, d_i(n)) = (0, \cdots, i-1, i+1, \cdots, n+1)$,

$$s_i \in \Delta([n+1],[n]) \quad s_i(j) = \begin{cases} j \text{ if } j \leq i \\ j-1 \text{ if } j > i \end{cases}$$

i.e. $s_i(0, \cdots, n+1) = (s_i(0), \cdots, s_i(n+1)) = (0, \cdots, i-1, i, i, i+1, \cdots, n)$.

Definition 12 (cosimplicial relations)**.** The maps $(d_i)_{i\in\mathbb{N}}$ and $(s_i)_{i\in\mathbb{N}}$ follow the **cosimplicial relations** defined by:

$$\begin{aligned}
&d_j d_i = d_i d_{j-1} \text{ if } i < j \\
&s_j d_i = d_i s_{j-1} \text{ if } i < j \\
&s_j d_j = id = s_j d_{j+1} \\
&s_j d_i = d_{i-1} s_j \text{ if } i > j+1 \\
&s_j s_i = s_i s_{j+1} \text{ if } i \leq j.
\end{aligned}$$

Proposition 2. *Every map $f \in \Delta^{\mathbf{op}}([n],[m])$ has a unique factorization of the form $f = d_{i_1} \cdots d_{i_k} s_{j_i} \cdots s_{j_l}$ satisfying $n \geq i_1 > \cdots > i_k \geq 0, \quad 0 \leq j_1 < \cdots < j_l \leq m$ and $n + l - k = m$.*

Proof. The proof for the existence is a computation, and the uniqueness is due to the cosimplicial relations. □

Definition 13 (Simplicial set)**.** A **simplicial set** is a functor $X : \Delta^{\mathbf{op}} \to \mathbf{Set}$. We write $\mathbf{sSet} = \mathbf{Fun}(\Delta^{\mathbf{op}}, \mathbf{Set})$. Given a simplicial set X, we write $X_n = X[n]$.

Remark 6. *Thanks to the coface and codegeracy maps, when we have a simplicial sets, we only need to know the images of the objects of $|\Delta|$ and on the coface and codegeneracy maps. With the factorization we can deduce how it works on all the maps of Δ, i.e. if X is a singular set, and $\theta \in \Delta^{\mathbf{op}}([n],[m])$ such that $\theta = d_{i_1} \cdots d_{i_k} s_{j_i} \cdots s_{j_l}$ verifying the last remark, $X\theta = X(d_{i_1} \cdots d_{i_k} s_{j_i} \cdots s_{j_l}) = X(s_{j_l}) \cdots X(s_{j_i}) X(d_{i_k}) \cdots X(d_{i_1})$. This leads us to a second definition of the simplicial sets.*

Definition 14 (Simplicial set)**.** A **simplicial set** X is a collection of objects $(X_n)_{n\in\mathbb{N}}$, and for each $n \in \mathbb{N}$ functions $D_i : X_{n+1} \to X_n$ and $S_i : X_n \to X_{n+1}$ for each $0 \leq i \leq n$ such that they follow the **simplicial relations**:

$$\begin{aligned} D_iD_j &= D_{j-1}D_i \text{ if } i < j \\ D_iS_j &= S_{j-1}D_i \text{ if } i < j \\ D_jS_j &= id = D_{j+1}S_j \\ D_iS_j &= S_jD_{i-1} \text{ if } i > j+1 \\ S_iS_j &= S_{j+1}S_i \text{ if } i \leq j. \end{aligned}$$

Proposition 3. *The two definitions of simplicial sets are equivalent.*

Proof. This holds from the proposition 2. □

Remark 7. *Starting from a simplicial complex X, we can build a simplicial set. We put in X_n not only all n-simplices of X, but also all the degenerated i-simplices with $i < n$ so X_0 will be all the vertices, X_1 the lines and degenerated vertices, X_2 the triangles, and degenerated lines and vertices. $(D_i)_{i\in\mathbb{N}}$ correspond to face maps that assigns a simplex to one of its faces, and $(S_i)_{i\in\mathbb{N}}$ are degeneracy maps that transform a simplex in a degenerated simplex of higher dimension.*

Definition 15 (Simplicial morphism)**.** Morphisms in **sSet** are called **simplicial morphisms**. They correspond to natural transformations between contravariant functors from Δ to **Set**.

Definition 16. We call $\Delta : \Delta \to \mathbf{sSet} \quad \Delta[n] = \Delta(-,[n])$ the covariant Yoneda embedding functor.

We can now proceed to an application of the Yoneda Lemma.

Proposition 4. *Given a simplicial set X, we have*

$$\mathbf{sSet}(\Delta[n], X) \quad \cong \quad X_n.$$

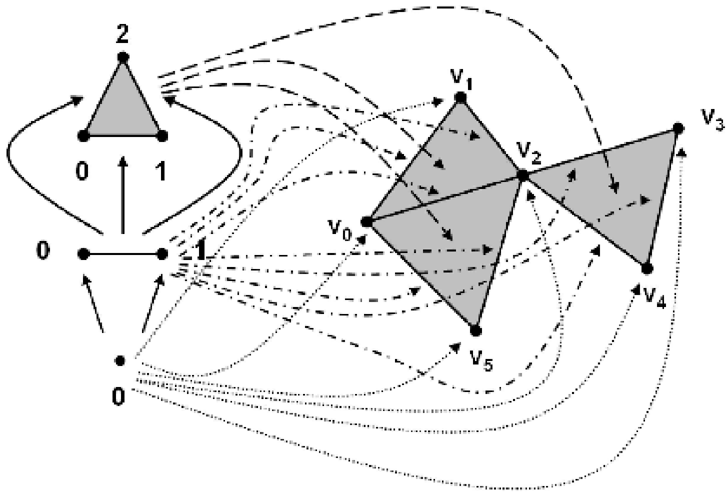

Figure 3: A simplicial complex as the image of ordered simplices.

Proof. We have $\mathbf{sSet}(\Delta[n], X) = \mathbf{Nat}(\Delta[n], X)$, thus the theorem is just the application of the Yoneda Lemma. □

Remark 8. *This proposition allows us to think to the nth component of X as "the set of n-simplexes".*

4.3 Singular sets

Singular sets constitute an important example of how the singular sets are used in topology.

Definition 17 (Standard n-simplex)**.** We write $|\Delta^n|$ the **standard n-simplex** defined by $|\Delta^n| = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \forall i\}$ with the induced standard topology.

Definition 18 (Singular set)**.** Let Y be a topological space. The **singular set** $\mathcal{S}(Y)$ is a functor $\Delta^{\mathbf{op}} \to \mathbf{Set}$ (so a singular set) such that

$$\mathcal{S}(Y)[n] = \mathbf{Top}(|\Delta^n|, Y)$$

i.e. $\mathcal{S}(Y)[n]$ is the set of all the continuous maps from $|\Delta^n|$ to Y.
D_i and S_i are defined by: if $\sigma \in \mathcal{S}(Y)[n]$, $D_i\sigma = \sigma d_i : |\Delta^{n-1}| \to Y$, $S_i\sigma = \sigma s_i : |\Delta^{n+1}| \to Y$

$$\begin{array}{ccc} [n] & & \mathbf{Top}(|\Delta^n|, Y) \\ d_i \uparrow & \overset{\mathcal{S}(Y)}{\Rightarrow} & D_i \downarrow \\ [n\text{-}1] & & \mathbf{Top}(|\Delta^{n-1}|, Y) \end{array} \qquad \begin{array}{ccc} [n] & & \mathbf{Top}(|\Delta^n|, Y) \\ s_i \uparrow & \overset{\mathcal{S}(Y)}{\Rightarrow} & S_i \downarrow \\ [n\text{-}1] & & \mathbf{Top}(|\Delta^{n-1}|, Y) \end{array}$$

With that we can transpose our standard simplicial complexes in topological spaces, we also want to create simplicial complexes from simplicial sets, so we introduce a new functor.

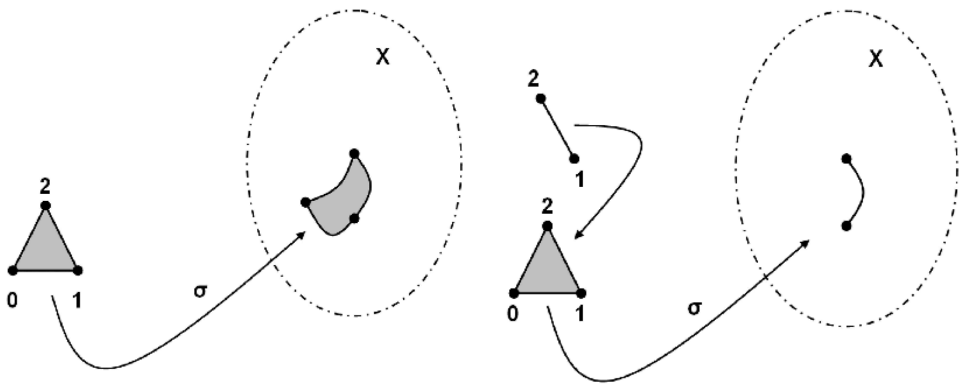

Figure 4: A singular simplex (left) and one of its faces (right).

Definition 19 (Realization functor)**.** The **realization functor** $|-| : \mathbf{sSet} \to \mathbf{Top}$ is defined as follows:

$$|X| = \left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n| \right) / \sim,$$

where $\sim$ is the equivalence relation generated by $(x, d_i(p)) \sim (D_i(x), p)$ for $x \in X_{n+1},\ p \in |\Delta^n|$ and $(x, s_i(p)) \sim (S_i(x), p)$ for $x \in X_{n-1},\ p \in |\Delta^n|$.

This definition doesn't look immediate. What it does is taking a simplicial set X and build a simplicial complex, so particularly a topological space. The idea is that for each element of X_n we associate an n-simplex, so we have a collection of disjoints simplexes $\coprod_{n\in\mathbb{N}} X_n \times |\Delta^n|$. The aim of the relation $\sim$ is to glue these simplices according to the information given by X. For example, it takes $(D_i(x), p)$ which is the *ith* face of x considered as a n-simplex, and identifies it with the *ith* face of x represented by $(x, d_i(p))$, a $(n+1)$-simplex. It glues each face of x considered as stand-alone simplices to the corresponding face of x. Since the identification is also done for any other $x,\ y$ such that $D_j(y) = D_i(x)$, if two simplices share the same face, it will only be counted once. Similarly, we suppress the degenerate simplices, since they already appear as nondegenerate simplices.

Remark 9. *$|\Delta[n]|$ is homeomorphic to $|\Delta^n|$.*

Theorem 3. *Given a topological space Y and a simplicial set X, we have the following bijection:*

$$\mathbf{Top}(|X|, Y) \quad \cong \quad \mathbf{sSet}(X, \mathcal{S}(Y)).$$

Proof. The proof builds explicitely the bijective functions (see [3] proposition 4.10). □

This is very strong, each continuous function we have from the simplicial complex to Y can be seen as a natural transformation between two simplicial sets, which can be studied with the standard simplices thanks to the Yoneda Lemma. This construction took some time and efforts but it provides a combinatorical model for the homotopy theory of topological spaces.

5 Conclusion

Since category theory is a very abstract field of mathematics, it takes time to get used to the vocabulary and be able to "see" what is happening. Nevertheless, this paper covers the key elements and basic examples towards a better understanding and the intention behind the last part was to give a more visual way to represent it. We hope we managed to spark interest in the reader to this very recent and still expanding field of mathematics.

References

[1] Page wikipedia - simplicial set. `http://en.wikipedia.org/wiki/Simplicial_set`.

[2] Francis Borceux. *Handbook of Categorical Algebra : Volume 1, Basic Category Theory.* Cambridge University Press, 1994.

[3] Greg Friedman. An elemtary illustrated introduction to simplicial sets. December 2011. arXiv:0809.4221v2.

[4] Saunders Mac Lane. *Categories for the working mathematician, Second Edition.* Springer-Verlag, 1971.

[5] John F. Jardine Paul G. Goerss. *Simplicial homotopy theory.* Birkhäuser, 2009.

[6] Emily Riehl. A leisurely introduction to simplicial sets. August 2011.

[7] Andrew Snowden. An introduction to simplicial sets. Spring 2011.